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Atomic parity-violation and precision physics

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Abstract. The atomic parity-violation (APV) parameter Q_{W} for a nucleus with n neutrons and z protons has been included in the list of pseudo-observables accessible with the codes TOPAZ0 and ZFITTER. In this way one can add the APV results in the LEP EWWG "global" electroweak fits, checking the corresponding effect when added to the existing precision measurements.

1 Introduction

Recently we were asked to include atomic parity-violation (hereafter APV) parameters in the list of pseudo-observables (hereafter PO) that are accessible with the FORTRAN codes TOPAZ0 [1] and ZFITTER [2], so as to include the APV results in the LEP EWWG electroweak fits.

The reason for this operation is that there are now precise experiments measuring APV in cesium [3], at the 0.4% level, thallium [4], lead [5] and bismuth [6]. Moreover, according to [7], the uncertainties associated with the atomic wave functions have been reduced to another 0.4% for cesium. For additional uncertainties associated with the value of the tensor polarizability we refer to [8]. Note however that there is an intrinsic difference between the PO at the Z resonance, e.g. $\mathit{\Gamma}_Z$, σ_h^0, A_{FB}^0 etc, and the APV parameters where the typical scale is dictated by the limit of zero momentum transfer in the APV Hamiltonian. This fact alone is the origin of a comparatively larger theoretical uncertainty which is due to our basic ignorance of QCD corrections in this regime.

The investigation of APV has been the subject of a number of studies made in the 80's by Marciano and Sirlin [9] and [10]. For TOPAZ0, which is based on the generalized minimal subtraction scheme [11], it has been relatively simple to include all recently computed higher-order effects in the old $\overline{\mathrm{MS}}$ calculation. For ZFITTER instead, the authors have been able to produce a novel evaluation of the APV parameters in the on mass-shell (OMS) scheme. The current value for the weak charge is

$$Q_{\mathrm{W}}(\mathrm{Cs}) = -72.06 \pm 0.28 \pm 0.34\,(\mathrm{theo.}) \tag{1}$$

For a recent evaluation of Q_{W} we refer, again, to [8] where the program GAPP [12] has been used.

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2 Upgrading the $\overline{\mathrm{MS}}$ calculation

The electron–quark parity-violating Hamiltonian at zero momentum transfer will be conventionally parameterized as follows:

$$\begin{aligned} H_{\mathrm{PV}} = \frac{G_{\mathrm{F}}}{\sqrt{2}}(&C_{1u}\bar{e}\gamma_\mu\gamma_5 e\bar{u}\gamma_\mu u + C_{2u}\bar{e}\gamma_\mu e\bar{u}\gamma_\mu\gamma_5 u \\ &+C_{1d}\bar{e}\gamma_\mu\gamma_5 e\bar{d}\gamma_\mu d + C_{2d}\bar{e}\gamma_\mu e\bar{d}\gamma_\mu\gamma_5 d)\ldots, \end{aligned} \tag{2}$$

where the ellipsis represents heavy quark terms and we have factorized out the Fermi constant G_{F}. In heavy atoms the dominant part of parity violation is proportional to the so-called weak charge Q_{W}

$$Q_{\mathrm{W}}(Z,A) = 2[(Z+A)\,C_{1u} + (2A-Z)\,C_{1d}]\,. \tag{3}$$

We have taken the calculation by Marciano and Sirlin which is performed in the modified minimal subtraction scheme ($\overline{\mathrm{MS}}$) and have extended it to include all higher-order effects presently known. To summarize: the two-loop leading contribution for the ρ-parameter [13], exact $\mathcal{O}\left(\alpha\alpha_{\mathrm{S}}\right)$ corrections [14], $\mathcal{O}\left(\alpha\alpha_{\mathrm{S}}^2\right)$ corrections to ρ [15], next-to-leading two-loop heavy top corrections [16]. At the same time, an attempt has been made to evaluate the theoretical uncertainty at the level of electroweak and of QCD corrections.

TOPAZ0 now returns, among all PO, the two quantities, C_{1u} and C_{1d} of (2). They are defined as follows:

$$\begin{aligned} C_{1u} &= -\frac{1}{2}\rho'_{\mathrm{PV}}\left[1-\frac{8}{3}\kappa'_{\mathrm{PV}}(0)\sin^2\hat{\theta}(M_W^2)\right], \\ C_{1d} &= \frac{1}{2}\rho'_{\mathrm{PV}}\left[1-\frac{4}{3}\kappa'_{\mathrm{PV}}(0)\sin^2\hat{\theta}(M_W^2)\right], \end{aligned} \tag{4}$$

where $\sin^2\hat{\theta}(M_W^2)$ is the $\overline{\mathrm{MS}}$ weak-mixing angle at the scale $\mu = M_W$.

We adopt a specific implementation of the re-summation procedure where the pair M_W and $\sin^2\hat{\theta}(M_W^2)$ is the solution of a system of coupled non-linear equations that include all available higher-order effects, as described in Sect. 6.11 and 8 of [17]. Moreover,

$$\begin{aligned}\rho'_{\mathrm{PV}} &= \rho - \frac{\alpha}{2\pi}\left[1+\frac{1}{\hat{s}^2}+4\hat{v}_e B_{\mathrm{p(np)}}\right.\\&\left.+\frac{9}{16\hat{s}^2\hat{c}^2}\left(1-\frac{16}{9}\hat{s}^2\right)\left(1+\hat{v}_e^2\right)\right],\\ \kappa'_{\mathrm{PV}}(0) &= \kappa_{\mathrm{PV}}(0)-\frac{\alpha}{2\pi\hat{s}^2}\left[\frac{9-8\hat{s}^2}{8\hat{s}^2}-\frac{\hat{v}_e}{6}\left(\ln\frac{M_Z^2}{m_e^2}+\frac{1}{6}\right)\right.\\&+\left(\frac{9}{4}-4\hat{s}^2\right)\hat{v}_e B_{\mathrm{p(np)}}\\&\left.+\frac{9}{16\hat{s}^2\hat{c}^2}\left(\frac{1}{2}\hat{v}_e+\frac{16}{9}\hat{s}^4\right)\left(1+\hat{v}_e^2\right)\right],\end{aligned}\tag{5}$$

where $\hat{v}_e = 1-4\hat{s}^2$, $\hat{s}^2=\sin^2\hat{\theta}(M_W^2)$ and where we have two different treatments of the Z–γ boxes – perturbative [9]

$$B_{\mathrm{p}} = \ln\frac{M_Z^2}{m^2}+\frac{3}{2},\quad m=m_u=m_d=75\,\mathrm{MeV},\tag{6}$$

and non-perturbative [10]

$$\begin{aligned}B_{\mathrm{np}} &= K+\frac{4}{5}\,(\xi_1)_B^p\,,\\ K &= M_Z^2\int_{M^2}^{\infty}\frac{\mathrm{d}u}{u(u+M_Z^2)}\left[1-\frac{\alpha_{\mathrm{S}}(u)}{\pi}\right],\\ (\xi_1)_B^p &= 2.55,\end{aligned}\tag{7}$$

where M is a mass scale representing the onset of the asymptotic behavior, i.e. the regime where α_{S} becomes small. We observe a plateau of stability in K for M centered around 0.5 GeV and this is the numerical value used. Furthermore we used the following form for the ρ,κ parameters [9]:

$$\begin{aligned}\rho &= 1+\frac{\alpha}{4\pi\hat{s}^2}\left[\frac{3}{4\hat{s}^2}\ln\hat{c}^2-\frac{7}{4}\right.\\&+\frac{3}{4}\frac{m_t^2}{M_W^2}\left(1+\delta_{\mathrm{EW}}+\delta_{\mathrm{QCD}}\right)\\&\left.+\frac{3}{4}h\left(\frac{\ln(\hat{c}^2/h)}{\hat{c}^2-h}+\frac{1}{\hat{c}^2}\frac{\ln h}{1-h}\right)\right],\\ \kappa_{\mathrm{PV}}(0) &= 1-\frac{\alpha}{2\pi\hat{s}^2}\\&\times\left[\frac{7}{9}-\frac{\hat{s}^2}{3}+\frac{Q_f}{3}\sum_f\left(I_e^{(3)}-2Q_f\hat{s}^2\right)\ln\frac{m_f^2}{M_W^2}\right].\end{aligned}\tag{8}$$

In the previous equation we have $h = M_H^2/M_Z^2$. The strange quark mass is effectively chosen to be $m_s = 250\,\mathrm{MeV}$ so that (with effective $m_u = m_d = 75\,\mathrm{MeV}$) we recover the dispersive analysis for the Z–γ transition where $\Pi_{\mathrm{Z}\gamma}(0)$ is rewritten in terms of a dispersion relation with the kernel connected to $\sigma(e^+e^-\to\mathrm{hadrons})$. In (8) the $\delta_{\mathrm{EW\ (QCD)}}$ are the LO+NLO electroweak ($\mathcal{O}\left(\alpha_{\mathrm{S}}^2+\alpha_{\mathrm{S}}^3\right)$ QCD) correction to ρ. The evaluation of ρ and $\hat{s}^2$ includes the best available LO+NLO terms [17].

TOPAZ0 default is the perturbative formulation of the *factorized* result of (4). There is the option of using some *additive* formulation where

$$\begin{aligned}C_{1u} &= -\frac{1}{2}\rho\left[1-\frac{8}{3}\kappa_{\mathrm{PV}}(0)\sin^2\hat{\theta}(M_W^2)\right]+\Delta_u,\\ C_{1d} &= \frac{1}{2}\rho\left[1-\frac{4}{3}\kappa_{\mathrm{PV}}(0)\sin^2\hat{\theta}(M_W^2)\right]+\Delta_d,\end{aligned}\tag{9}$$

where $\Delta_{u,d}$ are obtained from (4) by expanding and by neglecting terms of $\mathcal{O}\left(\alpha^2\right)$.

3 Atomic parity violation in OMS scheme

The old result of [9] has been completely re-derived in the OMS scheme. Here, the technical problem is represented by the extraction of the limit of zero momentum transfer from the expressions that have been derived for the process $ee\to t\bar{t}$ [18]. Here the process under consideration is the t channel scattering $ee\to uu$ and what we need is naturally contained in the results of [18] since they were derived retaining all masses and, therefore, the limit of zero momentum transfer, $Q^2 << \text{(all)}\ m^2$, is possible.

It is rather easy to take the limit $Q^2\to 0$ for vertices and self-energy functions since they depend only on this variable. For boxes the procedure is more complex due to their complicated dependence on s and t invariants. Fortunately enough, ZZ and $Z\gamma$ boxes form a gauge invariant subset of the whole result and for WW boxes one has to replace, in the corresponding limit, only the $\xi=1$ part of the result, which is well defined and simple. This fact triggered the strategy for a calculation where we take all contributions but boxes from the $Q^2\to 0$ limit of the $ee\to t\bar{t}$ form factors and were we have re-computed, from scratch, box diagrams at $Q^2=0$. Note that this calculation was done with the aid of the computer system described in [19].

For our calculation we compare the APV Hamiltonian of (2) with its $ee\to t\bar{t}$ analog, (I.10) of [18]:

$$\begin{aligned}\mathcal{A}_{\mathrm{Z}}(0) &= I_e^{(3)}I_f^{(3)}\frac{\pi\alpha}{s_W^2c_W^2(-M_Z^2)}\\&\times\Big\{\gamma_\mu\gamma_+\otimes\gamma_\mu\gamma_+F_{\mathrm{LL}}(0)+d_e\gamma_\mu\otimes\gamma_\mu\gamma_+F_{\mathrm{QL}}(0)\\&+d_f\gamma_\mu\gamma_+\otimes\gamma_\mu F_{\mathrm{LQ}}(0)+d_ed_f\gamma_\mu\otimes\gamma_\mu F_{\mathrm{QQ}}(0)\Big\}.\end{aligned}\tag{10}$$

Here (0) stands for $Q^2=0$ and we write only one argument since box contributions are excluded. Moreover,

$$\gamma_+ = 1+\gamma_5,\quad d_f=-4|Q_f|s_W^2.\tag{11}$$

From (2) and (10) we immediately derive a relation between the APV parameters C_{1f} and C_{2f} and the $ee\to t\bar{t}$

form factors at zero momentum transfer:

$$C_{1f} = I_f^{(3)}[f_{\rm LL} + d_f f_{\rm LQ} - \Delta r\,(1+d_f)],$$
$$C_{2f} = I_f^{(3)}[f_{\rm LL} + d_e f_{\rm QL} - \Delta r\,(1+d_e)]. \tag{12}$$

Here $f = u, d$ and

$$f_{\rm LL,QL,LQ} = 1 + \frac{\alpha}{4\pi s_W^2} F_{\rm LL,QL,LQ}(0). \tag{13}$$

After lengthy but straightforward calculations, we are able to reproduce the following generic expressions:

$$\begin{aligned} C_{1u} = &-2I_e^{(3)}\rho_{\rm PV}\left(I_u^{(3)} - 2Q_u\kappa_{\rm PV}s_W^2\right) \\ &+\frac{\alpha}{\pi}\left[Q_e^2 a_e v_u + \frac{1}{3}Q_u Q_\nu\left(\ln r_{{\rm W}e} + \frac{1}{6}\right)\right. \\ &+\frac{2}{3}Q_u Q_e v_e a_e\left(\ln r_{{\rm Z}e} + \frac{1}{6}\right) \\ &+C_f^{\rm WW} + 3Q_u a_u Q_e v_e\left(\ln r_{{\rm Z}u} + \frac{3}{2}\right) \\ &\left.+\frac{3}{4s_W^2 c_W^2} v_u a_u\left(v_e^2 + a_e^2\right)\right], \end{aligned} \tag{14}$$

where $v_f = I_f^{(3)} - 2Q_f s_W^2$, $a_f = I_f^{(3)}$ are the usual vector and axial-vector couplings and we introduced the notation $r_{ij} = m_i^2/m_j^2$ and a fictitious term with non-zero neutrino charge in order to have a completely general representation and where

$$C_f^{\rm WW} = \begin{cases} \dfrac{1}{2s_W^2} & \text{for} \quad f = u, \\ -\dfrac{1}{8s_W^2} & \text{for} \quad f = d, \end{cases} \tag{15}$$

is a contribution, originating from the WW box, which is different for u and d channels (direct–crossed).

The other APV parameters can be obtained with the aid of some simple substitutions:

$$C_{2u} = C_{1u}|_{e\,\leftrightarrow\,u,\,Q_\nu\,\to\,Q_d},$$
$$C_{1(2)d} = C_{1(2)u}|_{u\,\leftrightarrow\,d}. \tag{16}$$

The terms of $\mathcal{O}(\alpha/\pi)$ in (14) are identical to corresponding terms of the $\overline{\rm MS}$ result. In the sequential order they are due to the QED vertex in Z exchange, the W abelian vertex in γ exchange (with neutrino charge), the Z abelian vertex in γ exchange; the WW, $Z\gamma$ and ZZ boxes.

The only difference with respect to the $\overline{\rm MS}$ result is present in the first term. The factor ρ is almost the same:

$$\begin{aligned} \rho_{\rm PV} = 1 + \frac{\alpha}{4\pi s_W^2}&\left\{\frac{3}{4}\left[-\frac{1}{s_W^2}\ln c_W^2 - \frac{r_{\rm HW}}{1-r_{\rm HW}}\ln r_{\rm HW}\right.\right. \\ &\left.\left.+\frac{r_{\rm HW}}{1-r_{\rm HZ}}\ln r_{\rm HZ}\right] - \frac{7}{4} - \Delta\rho^{\rm fer}(0)\right\}, \end{aligned} \tag{17}$$

where we use instead the full expression for $\rho^{\rm fer}(0)$,

$$\Delta\rho^{\rm fer}(0) = \frac{\Sigma_{WW}^{\rm fer}(0) - \Sigma_{ZZ}^{\rm fer}(0)}{M_W^2}, \tag{18}$$

contrary to the approximation made above where only the (leading) quadratic term in m_t is retained. The difference, being proportional to light fermion masses, is numerically rather small.

However, the main difference with the $\overline{\rm MS}$ calculations is confined in the APV parameter $\kappa_{\rm PV}$ for which, in the OMS scheme, we derived

$$\begin{aligned} \kappa_{\rm PV} = 1 + \frac{\alpha}{4\pi s_W^2}&\left\{\left(\frac{1}{6} + 7c_W^2\right)L_\mu(M_W^2) - \frac{8}{9} - \frac{2}{3}c_W^2\right. \\ &\left.-\frac{c_W^2}{s_W^2}\left(\Delta\rho^{{\rm bos},F} + \Delta\rho^{{\rm fer},F}\right) - \Pi_{Z\gamma}^{\rm fer}(0)\right\}, \end{aligned} \tag{19}$$

where

$$L_\mu(M_W^2) = \ln\frac{M_W^2}{\mu^2}.$$

The gauge invariant Veltman $\Delta\rho$ parameter is

$$\Delta\rho = \frac{\Sigma_{WW}^{\rm fer}(M_W^2) - \Sigma_{ZZ}^{\rm fer}(M_Z^2)}{M_W^2}, \tag{20}$$

and contains both the *bosonic* and the *fermionic* components. We explicitly give the bosonic part, $\Delta\rho^{\rm bos}$ (for the definition of finite part $B_0^{\rm F}$ of B_0 functions see [17]):

$$\begin{aligned} \Delta\rho^{{\rm bos},F} = &\left(\frac{1}{12c_W^4} + \frac{4}{3c_W^2} - \frac{17}{3} - 4c_W^2\right) \\ &\times\left[B_0^{\rm F}\left(-M_W^2;M_W,M_Z\right) - c_W^2 B_0^{\rm F}\left(-M_Z^2;M_W,M_W\right)\right] \\ &+\left(1 - \frac{1}{3}r_{\rm HW} + \frac{1}{12}r_{\rm HW}^2\right)B_0^{\rm F}\left(-M_W^2;M_W,M_H\right) \\ &-\left(1 - \frac{1}{3}r_{\rm HZ} + \frac{1}{12}r_{\rm HZ}^2\right)\frac{1}{c_W^2}B_0^{\rm F}\left(-M_Z^2;M_Z,M_H\right) \\ &-4s_W^2 B_0^{\rm F}\left(-M_W^2;M_W,0\right) \\ &+\frac{1}{12}\left[\left(\frac{1}{c_W^4} + \frac{6}{c_W^2} - 24 + r_{\rm HW}\right)L_\mu(M_Z^2)\right. \\ &+s_W^2 r_{\rm HW}^2\left[L_\mu(M_H^2) - 1\right] \\ &-\left(\frac{1}{c_W^2} + 14 + 16c_W^2 - 48c_W^4 + r_{\rm HW}\right)L_\mu(M_W^2) \\ &\left.-\frac{1}{c_W^4} - \frac{19}{3c_W^2} + \frac{22}{3}\right]. \end{aligned} \tag{21}$$

To establish a link with the $\overline{\rm MS}$ calculation we introduce the usual notion of leading and remainder terms:

$$\kappa_{\rm PV} = 1 + \frac{\alpha}{4\pi s_W^2}\{(\Delta\kappa_{\rm PV})_{\rm lead} + (\Delta\kappa_{\rm PV})_{\rm rem}\}, \tag{22}$$

where the leading term contains only $\Delta\rho$ and the remainder contains all the rest:

$$(\Delta\kappa_{\rm PV})_{\rm lead} = -\frac{c_W^2}{s_W^2}\left(\Delta\rho^{{\rm bos},F} + \Delta\rho^{{\rm fer},F}\right)|_{\mu=M_W},$$
$$(\Delta\kappa_{\rm PV})_{\rm rem} = -\frac{8}{9} - \frac{2}{3}c_W^2 - \Pi_{Z\gamma}^{\rm fer}(0)|_{\mu=M_W}. \tag{23}$$

Numerically, $(\Delta\kappa_{\rm PV})_{\rm lead}$ and $(\Delta\kappa_{\rm PV})_{\rm rem}$ are nearly equal and one might think that the usual leading–remainder

Table 1. Predictions for $Q_{\rm W}({\rm Cs})$ from TOPAZ0 and ZFITTER for $M_Z = 91.1875\,{\rm GeV}$, $M_H = 150\,{\rm GeV}$ and $\alpha_{\rm S}(M_Z^2) = 0.119$

m_t [GeV]	170	175	180
Fact/Pert	−72.9712	−72.9632	−72.9551
Fact/Non Pert	−73.1994	−73.1932	−73.1869
Add/Pert	−72.9732	−72.9658	−72.9582
ZFITTER	−72.9762	−72.9698	−72.9637
Add/Non Pert	−73.2026	−73.1969	−73.1912

splitting, the standard factorization of contributions with different scales and re-summation (see [17]),

$$\kappa_{\rm PV} = \left[1 + \frac{\alpha}{4\pi s_W^2}\left(\Delta\kappa_{\rm PV}\right)_{\rm lead}\right] \times \left[1 + f_c\frac{\alpha}{4\pi s_W^2}\left(\Delta\kappa_{\rm PV}\right)_{\rm rem}\right], \quad (24)$$

with a conversion factor

$$f_c = \frac{\sqrt{2}G_\mu M_Z^2 s_W^2 c_W^2}{\pi\alpha}, \quad (25)$$

is not too well justified for the APV parameter $\kappa_{\rm PV}$. Note, however, that the factorized form of (24) is fully consistent with the $\overline{\rm MS}$ result (8) if we identify

$$\sin^2\hat\theta_{\rm W}(M_W) = \left[1 - f_c\frac{\alpha}{4\pi}\,\frac{c_W^2}{s_W^4}\Delta\rho^F\Big|_{\mu=M_W}\right] s_W^2. \quad (26)$$

As done before, for $\Pi^{\rm fer}_{\rm Z\gamma}(0)$ we use effective quark masses which are consistent with a dispersive treatment of $\Pi^{\rm fer}_{\rm Z\gamma}$ at zero scale.

Finally, we apply mixed QCD $\left(\mathcal{O}\left(\alpha_{\rm S}^2 + \alpha_{\rm S}^3\right)\right)$ and LO+ NLO electroweak two-loop corrections for the Veltman $\Delta\rho$ parameter. For $\rho_{\rm PV}$ we stick with one-loop (non-resummed) result (17), since there the notion of leading–remainder splitting fails completely (numerically it looks like $+3.5 - 3.0 = 0.5$). For the same reason, we apply to $\Delta\rho^{\rm fer}(0)$ only the mixed QCD but not the electroweak two-loop corrections, as already done in the first (8). The latter, as well as the other electroweak NLO corrections for the remainder terms, although not implemented, are successively used to evaluate the theoretical uncertainty in the electroweak sector of the OMS scheme.

4 Theoretical uncertainty in APV

In order to discuss the present level of theoretical uncertainty in atomic parity violation we start with the $\overline{\rm MS}$ results for $Q_{\rm W}({\rm Cs})$ that are shown in Table 1, corresponding to $M_Z = 91.1875\,{\rm GeV}$, $M_H = 150\,{\rm GeV}$ and $\alpha_{\rm S}(M_Z^2) = 0.119$. ZFITTER numbers corresponding to “Add/Pert” setup are added to the third row of the table.

The m_t dependence of $Q_{\rm W}({\rm Cs})$ is shown in Table 1 where we register a 0.22 (0.17) per-mill increase for m_t between 170 GeV and 180 GeV and for Pert (Non Pert). As for the M_H dependence we have computed a decrease of about 0.7 per-mill for M_H between 150 GeV and 300 GeV.

The associated theoretical uncertainty is approximately 3.2 per-mill and it is largely dominated by QCD effects. Let us consider the main sources of uncertainty. In the original calculation of Marciano and Sirlin we have a dependence of the result on light quark masses. This appearance can be seen in (5) and in the perturbative treatment of boxes, (6).

From 1983 the accuracy associated to the weak charge $Q_{\rm W}$ has been considerably reduced and we cannot include it in the list of high-precision PO if the result contains logarithmic enhancements due to light quark masses.

In their second paper Marciano and Sirlin [10] have suggested how to go beyond the partonic language. One should distinguish quark masses in the Z–γ transition and in Z–γ boxes. Light quark masses, m_{uds}, are then fixed to parameterize the dispersive result for the Z–γ transition and are not varied anymore in evaluating the theoretical uncertainty.

Furthermore we have Z–γ box diagrams where quark masses show up as a consequence of the zero momentum transfer limit. Here, according to the suggestion of [10] we split the boxes into a low-frequency part, approximated with the Born contribution for a physical nucleon (the $(\xi_1)^p_B$ term in (7)), and a high-frequency part (the K term in (7)) that includes $\mathcal{O}(\alpha_{\rm S})$ corrections where light quark masses disappear. The mass scale M separating low- from high-frequency parts is, of course, arbitrary and only subjected to the requirement that $\alpha_{\rm S}(Q^2)$ starts to become small for $|\,Q^2\,| > M^2$ and that $M > \Lambda_{\rm QCD}$. However, with the most complete evaluation of $\alpha_{\rm S}$ (up to three loops) we have found a plateau of stability for the result, i.e. for M between 0.5 and 0.6 (0.8) K goes from 9.2016 to 9.1737 (8.7818) and $Q_{\rm W}$ has a variation of 0.02 (0.3) per-mill. Therefore we fix $0.5 \le M \le 0.6$.

Instead of varying light quark masses between undefined limits we prefer to estimate the theoretical uncertainty by comparing the perturbative result with light quark masses fixed to reproduce the dispersive approach to the Z–γ transition with a non-perturbative ansatz based on a low-frequency high-frequency splitting at a mass scale of about 0.5 GeV. Note that when comparing $B_{\rm np}(M)$ with the perturbative factor $\ln M_Z^2/M^2 + (3/2)$ we find that the perturbative approach overestimates the effect of about 5.6% (1.9%) at $M = 0.5$ (0.8) GeV.

Furthermore, the differences in the factorized (4) versus additive (9) formulation of the coefficients $C_{1u,1d}$ is approximately 0.05 per-mill signaling that, from TOPAZ0’s treatment alone, pure electroweak higher orders are relatively under control. Another way of testing the electroweak theoretical uncertainty is, as usual, to compare two different renormalization schemes with the same input parameter set. When we compare ZFITTER in the preferred setup with TOPAZ0 additive/perturbative we obtain a relative difference of 0.04 (0.05, 0.08) per-mill at

$m_t = 170$ $(175, 180)$ GeV. However, an internal evaluation of electroweak theoretical uncertainties within ZFITTER (realized by evaluating the effect of the electroweak two-loop corrections which are not included in the preferred setup) shows a value of about ± 0.25 per-mill. Again, the conclusion is that theoretical uncertainty is completely dominated by QCD effects at zero momentum transfer.

Finally, let us define an effective APV weak-mixing angle by the following relation:

$$\sin^2\theta_{\mathrm{APV}} = \kappa'_{\mathrm{PV}}(0)\sin^2\hat{\theta}(M_W^2). \tag{27}$$

For $M_Z = 91.1875$ GeV, $M_H = 150$ GeV and $\alpha_{\mathrm{S}}(M_Z^2) = 0.119$ we obtain $\sin^2\theta_{\mathrm{APV}} = 0.231601$ (0.232123), corresponding to a perturbative (non-perturbative) treatment.

Appendix

A Taylor expansions

Here we list all expansions that are needed in order to reproduce the OMS results. Note that we need at most terms of $\mathcal{O}(s)$. In particular, in the calculation one needs several expansion of scalar three-point and two-point functions, usually termed C_0 and B_0:

$$\begin{aligned}
&C_0\left(-m_u^2, -m_u^2, -s; m_u, M, m_u\right) \\
&= \frac{1}{M^2}\Bigg\{-1 - \frac{7}{2}r_{u\mathrm{M}} - \frac{37}{3}r_{u\mathrm{M}}^2 \\
&\quad - \left(1 + 3r_{u\mathrm{M}} + 10r_{u\mathrm{M}}^2\right)\ln r_{u\mathrm{M}} \\
&\quad + \frac{s}{M^2}\left[\frac{1}{6r_{u\mathrm{M}}} + \frac{13}{12} + \frac{52}{9}r_{u\mathrm{M}} + \frac{673}{24}r_{u\mathrm{M}}^2 \right.\\
&\quad \left. + \left(\frac{1}{2} + \frac{10}{3}r_{u\mathrm{M}} + \frac{35}{2}r_{u\mathrm{M}}^2\right)\ln r_{u\mathrm{M}}\right]\Bigg\},
\end{aligned} \tag{28}$$

where $M = M_Z, M_H$ and we recall a short-hand notation for the mass ratios:

$$r_{ij} = \frac{m_i^2}{m_j^2}.$$

The other expansions read

$$\begin{aligned}
&C_0\left(-m_u^2, -m_u^2, -s; m_d, M_W, m_d\right) \\
&= \frac{1}{M_W^2}\Bigg\{-1 - r_{d\mathrm{W}} - r_{d\mathrm{W}}^2 - \left(\frac{5}{2} + 8r_{d\mathrm{W}}\right)r_{u\mathrm{W}} - \frac{10}{3}r_{u\mathrm{W}}^2 \\
&\quad -\left[1 + 2r_{d\mathrm{W}} + 3r_{d\mathrm{W}}^2 + (1 + 6r_{d\mathrm{W}})\,r_{u\mathrm{W}} + r_{u\mathrm{W}}^2\right]\ln r_{d\mathrm{W}} \\
&\quad + \frac{s}{M_W^2}\Bigg(\frac{1}{6}\left[\frac{1}{r_{d\mathrm{W}}} + \frac{11}{2} + 13r_{d\mathrm{W}} + \frac{47}{2}r_{d\mathrm{W}}^2 \right.\\
&\quad + \left(\frac{1}{r_{d\mathrm{W}}} + \frac{62}{3} + 97r_{d\mathrm{W}}\right)r_{u\mathrm{W}} + \left(\frac{1}{r_{d\mathrm{W}}} + \frac{187}{4}\right)r_{u\mathrm{W}}^2 \\
&\quad \left. + \frac{1}{r_{d\mathrm{W}}}r_{u\mathrm{W}}^3\right] + \left[\frac{1}{2} + 2r_{d\mathrm{W}} + 5r_{d\mathrm{W}}^2 \right.\\
&\quad \left. + \left(\frac{4}{3} + 10r_{d\mathrm{W}}\right)r_{u\mathrm{W}} + \frac{5}{2}r_{u\mathrm{W}}^2\right]\ln r_{d\mathrm{W}}\Bigg)\Bigg\},
\end{aligned} \tag{29}$$

$$\begin{aligned}
&C_0\left(-m_u^2, -m_u^2, -s; M_W, 0, M_W\right) \\
&= \frac{1}{M_W^2}\left[1 + \frac{r_{u\mathrm{W}}}{2} + \frac{r_{u\mathrm{W}}^2}{3} + \frac{s}{6M_W^2}\left(\frac{1}{2} + \frac{1}{3}r_{u\mathrm{W}}\right)\right],
\end{aligned} \tag{30}$$

$$C_0\left(0, 0, 0; M_H, 0, M_Z\right) = -\frac{1}{M_Z^2 - M_H^2}\ln r_{\mathrm{HZ}}, \tag{31}$$

$$\begin{aligned}
&B_0^{\mathrm{F}}\left(-m_u^2; M_W, 0\right) \\
&= -L_\mu(M_W^2) + 1 + \frac{1}{2}r_{u\mathrm{W}} + \frac{1}{6}r_{u\mathrm{W}}^2,
\end{aligned} \tag{32}$$

$$\begin{aligned}
&B_0^{\mathrm{F}}\left(-m_u^2; M, m_u\right) \\
&= -L_\mu(M^2) + \left(r_{u\mathrm{M}} + 2r_{u\mathrm{M}}^2\right)\ln r_{u\mathrm{M}} + 1 \\
&\qquad + \frac{1}{2}r_{u\mathrm{M}} + \frac{5}{3}r_{u\mathrm{M}}^2,
\end{aligned} \tag{33}$$

$$\begin{aligned}
&B_{0p}\left(-m_u^2; M, m_u\right) = -\frac{1}{2M^2}, \\
&B_0^{\mathrm{F}}\left(-s; m, m\right) = -L_\mu(m^2) + \frac{s}{6m^2},
\end{aligned} \tag{34}$$

$$\begin{aligned}
&B_0^{\mathrm{F}}\left(-s; m, M\right) \\
&= 1 + \left[M^2 L_\mu(M^2) - m^2 L_\mu(m^2)\right]\frac{1}{m^2 - M^2} \\
&\qquad + s\left[\frac{m^2 + M^2}{2(m^2 - M^2)^2} - \frac{m^2 M^2}{(m^2 - M^2)^3}\ln\left(\frac{m^2}{M^2}\right)\right].
\end{aligned} \tag{35}$$

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